ON A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS WITH NONUNIQUE SOLUTIONS

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1. INTRODUCTION

As we know, nonlinear differential equations subject to initial-value conditions possess unique solutions, under reasonable assumptions, whereas the same equations subject to two-point or multipoint boundary-value conditions can easily possess an infinite number of solutions. The purpose of this note is to describe a class of nonlinear differential equations subject to conditions analogous but not equivalent to multipoint conditions which can possess an infinite number of solutions. These equations are suggested by some work in respiratory control theory which will be described elsewhere.

2. A SCALAR EXAMPLE

Consider the equation

$$(2.1)\qquad \frac{du}{dt} = (a + bu(t_1))u(t), \quad u(0) = c,$$

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valid for $0 \leq t \leq t_2$, where $t_2 \geq t_1$. Using the explicit form of the solution of (2.1),

$$u(t) = ce^{(a+bu(t_1))t}, \tag{2.2}$$

we obtain the following transcendental equation for $u(t_1)$,

$$u(t_1) = ce^{at_1} e^{bt_1 u(t_1)}. \tag{2.3}$$

Depending upon the values of a, b, c, and t_1, this equation can have no, one, or two real roots, and it always possesses an infinite set of complex roots. To each root corresponds a solution of (2.1).

3. DISCUSSION

It is easy to see that the corresponding situation will persist for the general vector differential equation

$$\frac{dx}{dt} = g(x(t), x(t_1), \ldots, x(t_M)), \quad x(0) = c, \tag{3.1}$$

valid for $0 \leq t \leq t_0$, where $0 < t_1 < t_2 < \cdots < t_M < t_0$.

An interesting question is that of determining what additional conditions single out a unique solution.